ON LIAPUNOV STABILITY IN THE CRITICAL CASE OF A CHARACTERISTIC EQUATION WITH AN EVEN NUMBER OF ROOTS EQUAL TO ZERO

(OB USTOICHIVOSTI PO LIAPUNOVU V KRITICHESKOM SLUCHAE, KOGDA OPREDELIAIUSHCHEE URAVNENIE IMEET CHETNOE CHISLO NULEVYKH KORNEI)

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M.S. SAGITOV and A.N. FILATOV
(Tashkent)

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1. We shall consider the system of equations

$$\frac{dx_s}{dt} = p_{s1}x_1 + \ldots + p_{s,\,n+2m}x_{n+2m} + X_s(x_1, x_2, \ldots, x_{n+2m}) \quad \left(\begin{matrix} s = 1, \ldots, n + 2m \\ 2m < n \end{matrix}\right) \tag{1.1}$$

with assumption that the characteristic equation

$$\left|\begin{matrix} p_{11} - \lambda\, p_{12} & \cdots\, p_{1,\,n+2m} \\ \cdots & \cdots \\ p_{n+2m,\,1}\; p_2 \cdots & p_{n+2m,\,n+2m} - \lambda \end{matrix}\right| = 0 \tag{1.2}$$

has an even number $2m\ (m \geqslant 1)$ of roots equal to zero corresponding to which there are m groups of solution of the first approximation equations

$$\frac{dx_s}{dt} = p_{s1}x_1 + \ldots + p_{s,\,n+2m}x_{s,\,n+2m} \qquad (s = 1, \ldots, n + 2m) \tag{1.3}$$

We shall assume that the roots of Equation (1.2) which are not equal to zero have negative real parts, and that the $X_s\ (X_s(0, 0, \ldots, 0) \equiv 0)$ are holomorphic functions of the quantities $x_1, x_2, \ldots, x_{n+2m}$; the series expansions of these functions do not have any terms of order inferior to the second.

The purpose of the problem is the determination of the conditions for which the solution

$$x_1 = x_2 = \ldots = x_{n+2m} = 0 \tag{1.4}$$

of Equations (1.1) is stable or unstable, according to Liapinov. The case which occurs when Equation (1.2) has two roots equal to zero ($m = 1$) has been studied at length by Liapunov [1] and Kamenkov [2].

Kamenkov has also investigated in [2] the case in which the characteristic equation has: p roots equal to zero, to which correspond p groups of solutions, $2q$ purely imaginary roots, and also r roots with negative real parts (the sum $p + 2q + r$ is equal to the order of the system). We shall mention also that the case in which the characteristic equation has k roots equal to zero, corresponding to which there are $k - 1$ groups of solutions, and where k is the order of the system, has been considered in [3].

In this paper, following the ideas of [1], we study the case in which the characteristic equation (1.2) has any even number of roots equal to zero ($m > 1$). Some restrictions are imposed on the function X_s and are mentioned below.

2. For the assumptions made, and by means of a few constant coefficient linear transformations, the system (1.1) can be written as

$$\frac{dz_{2k-1}}{dt} = z_{2k} + Z_{2k-1}(z_1, \ldots, z_{2m};\ x_1, \ldots x_n)$$
$$\frac{dz_{2k}}{dt} = Z_{2k}(z_1, \ldots, z_{2m};\ x_1, \ldots, x_n) \qquad \begin{pmatrix} k = 1, \ldots, m \\ s = 1, \ldots, n \end{pmatrix} \tag{2.1}$$
$$\frac{dx_s}{dt} = p_{s1}x_1 + \ldots + p_{sn}x_n + X_s(z_1, \ldots, z_{2m};\ x_1, \ldots, x_n)$$

Without loss of generality, it can always be assumed (*) that $Z_{2k-1} \equiv 0$ and $X_s \equiv 0$, when $z_2 = z_4 = \ldots = z_{2m} = x_1 = x_2 = \ldots = x_n = 0$.

It is clear that the solution

$$z_1 = z_2 = \ldots = z_{2m} = x_1 = x_2 = \ldots = x_n = 0 \tag{2.2}$$

of the system (2.1) corresponds to the solution (1.4) of the system (1.1). We shall confine ourselves to the investigation of the stability of Equation (2.2) by assuming that the functions Z_{2k} and X_s have the form

$$Z_{2k} = \sum_{\mu=1}^{m} a_{2\mu}^{(2k)} z_{2\mu}^2 + \sum_{\mu=1}^{m} P_{2\mu}^{(2k)}(x_1, \ldots, x_n)\, z_{2\mu} + Q^{(2k)}(x_1, \ldots, x_n) +$$
$$+ \sum_{s=1}^{n} x_s \varphi_s^{(2k)}(z_1, z_3, \ldots, z_{2m-1}) + R^{(2k)}(z_1, \ldots, z_{2m};\ x_1, \ldots, x_n)$$
$$X_s = \sum_{i=1}^{n} x_i \varphi_{si}(z_1, z_3, \ldots, z_{2m-1}) + R_s(z_1, \ldots, z_{2m};\ x_1, \ldots, x_n)$$
$$(k = 1, \ldots, m; \quad s = 1, \ldots, n) \tag{2.3}$$

where the $a_{2\mu}^{(2k)}$ are constants, $P_{2\mu}^{(2k)}$ are linear forms of the variables $x_1, \ldots, x_n$; $\varphi_s^{(2k)}$, φ_{si} are holomorphic functions, cancelling themselves for $z_1 = z_3 = \ldots = z_{2m-1} = 0$; $Q^{(2k)}$ are quadratic forms of the quantities $x_1, \ldots, x_n$; $R^{(2k)}$, R_s are holomorphic functions of the variables $z_1, \ldots, z_{2m}$; $x_1, \ldots, x_n$, which do not include terms of these variables of order lower than the third.

We shall consider the function $\psi_s^{(2\mu)}(z_1, z_3, \ldots, z_{2m-1})$, determined from Equations

$$\sum_{s=1}^{n} (p_{si} + \varphi_{si})\psi_s^{(2\mu)} + \Big[1 + \Big(1 - \sum_{\nu=1}^{m} a_{2\mu}^{(2\nu)}\Big) z_{2\mu-1}\Big]\varphi_i^{(2\mu)} = 0 \tag{2.4}$$
$$(i = 1, \ldots, n; \quad \mu = 1, \ldots, m)$$

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*) This can be obtained by substituting

$$z_{2k} = \bar{z}_{2k} + \psi_{2k}(z_1, z_3, \ldots, z_{2m-1}), \qquad x_s = x_s + \varphi_s(z_1, z_3, \ldots, z_{2m-1})$$

where ψ_{2k} and φ_s satisfy Equations $(k = 1, \ldots, m,\ s = 1, \ldots, n)$

$$\psi_{2k} + Z_{2k-1}(z_1, z_3, \ldots, z_{2m-1};\ \psi_2, \psi_4, \ldots, \psi_{2m};\ \varphi_1, \varphi_2, \ldots, \varphi_n) = 0$$
$$p_{s1}x_1 + \ldots + p_{sn}x_n + X_s(z_1, z_3, \ldots, z_{2m-1};\ \psi_2, \psi_4, \ldots, \psi_{2m};\ \varphi_1, \varphi_2, \ldots, \varphi_n) = 0$$

It is evident that for every fixed μ, the system of equations (2.4) yields an unambiguous determination of the functions $\psi_s^{(2\mu)}$ $(s = 1, \ldots, n)$, which vanish for the values $z_1 = z_3 = \ldots = z_{2m-1} = 0$.

We shall now compose a Liapunov V function of the form

$$V = \sum_{\mu=1}^{m} \left[1 + \left(1 - \sum_{k=1}^{m} a_{2\mu}^{(2k)}\right) z_{2\mu-1}\right] z_{2\mu} +$$

$$+ \sum_{\mu=1}^{m} \sum_{k=1}^{m} U_{2\mu}^{(2k)} z_{2\mu} + \sum_{k=1}^{m} W^{(2k)} + \sum_{s=1}^{n} \sum_{k=1}^{m} x_s \psi_s^{(2k)} \tag{2.5}$$

Here $U_{2\mu}^{(2k)}$ and $W^{(2k)}$ are, respectively, linear and quadratic forms, determined by the equations (*)

$$\sum_{s=1}^{n} (p_{s1}x_1 + \ldots + p_{sn}x_n) \frac{\partial U_{2\mu}^{(2k)}}{\partial x_s} + P_{2\mu}^{(2k)} = -\sum_{s=1}^{n} x_s \left(\frac{\partial \psi_s^{(2k)}}{\partial z_{2\mu-1}}\right)_0 \tag{2.6}$$

$$\sum_{s=1}^{n} (p_{s1}x_1 + \cdots + p_{sn}x_n) \frac{\partial W^{(2k)}}{\partial x_s} + Q^{(2k)} = \sum_{s=1}^{n} x_s^2$$

Calculating the total derivative of the function V with respect to t, on the basis of Equation (2.1) and taking into consideration (2.3), (2.4) and (2.6), we get

$$\frac{dV}{dt} = \sum_{\mu-1}^{m} z_{2\mu}^2 + m \sum_{s=1}^{n} x_s^2 + S \tag{2.7}$$

$$S = \sum_{\mu=1}^{m} \left\{\left(1 - \sum_{k=1}^{m} a_{2\mu}^{(2k)}\right) z_{2\mu-1} \left[Q^{(2\mu)} + \sum_{\nu=1}^{m} \left(P_{2\nu}^{(2\mu)} z_{2\nu} + a_{2\nu}^{(2\mu)} z_{\nu 2}^2\right)\right]\right\} +$$

$$+ \sum_{\mu=1}^{m} \left[1 + \left(1 - \sum_{k=1}^{m} a_{2\mu}^{(2k)}\right) z_{2\mu-1}\right] R^{(2\mu)} + \sum_{k=1}^{m} \sum_{\mu=1}^{m} U_{2\mu}^{(2k)} Z_{2k} +$$

$$+ \sum_{\mu=1}^{m} \left(1 - \sum_{k=1}^{m} a_{2\mu}^{(2k)}\right) z_{2\mu} Z_{2\mu-1} + \sum_{k=1}^{m} \sum_{s=1}^{n} \frac{\partial W^{(2k)}}{\partial x_s} X_s + \tag{2.8}$$

$$+ \sum_{k=1}^{m} \sum_{\mu=1}^{m} \sum_{s=1}^{n} z_{2\mu} \frac{\partial U_{2\mu}^{(2k)}}{\partial x_s} X_s + \sum_{k=1}^{m} \sum_{\mu=1}^{m} \sum_{s=1}^{n} x_s \frac{\partial \psi_s^{(2k)}}{\partial z_{2\mu-1}} Z_{2\mu-1} +$$

$$+ \sum_{k=1}^{m} \sum_{s=1}^{n} \psi_s^{(2k)} R_s + \sum_{k=1}^{m} \sum_{\mu=1}^{m} \sum_{s=1}^{n} x_s z_{2\mu} \left[\frac{\partial \psi_s^{(2k)}}{\partial z_{2\mu-1}} - \left(\frac{\partial \psi_s^{(2k)}}{\partial z_{2\mu-1}}\right)_0\right]$$

From Equations (2.7) and (2.8) there follows that the function V satisfies the instability theorem of Chetaev [4]. Therefore, for the conditions given above, solution (2.2) and therefore solution (1.4) are unstable.

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*) The subscript zero, in the last term of the first equation of (2.6), denotes that the derivative is taken at the point $z_1 = z_3 = \ldots = z_{2m-1} = 0$

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